A new series of rotation invariant moments by Lie transformation group theory

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Abstract

In automatic word recognition, location-scale-rotation invariant features are important. We will consider rotation invariant moments of 2-D imagery. Two new series of invariant moments are derived by Lie theory of the orthogonal transformation group. The infinite series are expressed in terms of regular moments.

1. Introduction

In automatic word recognition, location-scale-rotation invariant features are important. One of such invariant features is an invariant moment, e.g. Hu's invariant moments [5-7] or Flusser invariant moments [2-4]. Shen and Ip [7] concisely summarized the essential idea of the derivation of rotation invariant moments including Zernike moments and Wavelet moments. Further, Diami [1] summarized beautifully the derivation method of invariants by Lie algebra of transformation groups related to the rotation invariant texture descriptor. In this paper we pursue the Lie algebra-based method and derive two new infinite series of location-scale-rotation invariant moments.

Section 2 reviews classical methods for deriving invariant moments proposed in the literature. The approach based on Lie algebra is also reviewed in Section 3. Using Lie theory, we derive two new series of rotation invariant moments in Section 4. The series are expressed in terms of regular moments and will be helpful in applications to word recognition. Section 5 concludes the paper.

2. Classical methods for rotation invariants

In this section we briefly review general construction methods of invariant moments summarized by Shen and Ip [7]. Let us define a regular moment $m_{p,q}$ of a binary or gray scale image $f(x,y)$ by

$$m_{p,q} = \iint x^p y^p f(x,y)dxdy \quad \text{for} \quad p,q = 0,1,2,... \tag{1}$$

and the centroid by

$$x_0 = m_{10} \quad \text{and} \quad y_0 = m_{01}. \tag{2}$$

By the coordinate change:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}, \tag{3}$$

the image $f(x - x_0, y - y_0)$ becomes location invariant. Transforming the image from $f(x,y)$ into $f(x,y)/m_{00}$, we have the standardized image in scale. Next we think of rotation invariance. For obtaining rotation invariant moments, Shen and Ip [7] used the following generalized moment expression based on the polar coordinate:

$$F_{p,q} = \iint f(r\cos\theta, r\sin\theta)g_p(r)\exp(\sqrt{-1}q\theta)rdrd\theta, \tag{4}$$

where $g_p(r)$ is a function of r with index p. Then, the following proposition holds.

Proposition 1 (Shen and Ip [7]). Let $F_{p,q}$ be the generalized moment defined by (4). Then, both of the norm $||F_{p,q}||$ and the combined moment $F_{p_1,q}\overline{F_{p_2,q}}$ are rotation invariant, where $\overline{z}$ means the complex conjugate of z.

Proof. Let $f^{rotated}(r\cos\theta, r\sin\theta)$ be a rotated image by an angle β. Then, it holds that $f^{rotated}(r\cos\theta, r\sin\theta)$ $= f(r\cos(\theta-\beta), r\sin(\theta-\beta))$, and

$$F^{rotated}_{p,q} = F_{p,q}e^{\sqrt{-1}q\beta}. \tag{5}$$

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The equation (5) leads the following identity:

$$||F_{p,q}^{rotated}||^2 = F_{p,q}^{rotated}\ \overline{F_{p,q}^{rotated}} = F_{p,q}\ \overline{F_{p,q}} = ||F_{p,q}||^2.$$

The same equation also yields the identity:

$$\begin{aligned} F_{p_1,q}^{rotated}\overline{F_{p_2,q}^{rotated}} &= F_{p_1,q}\overline{F_{p_2,q}}e^{\sqrt{-1}q\beta}e^{-\sqrt{-1}q\beta} \\ &= F_{p_1,q}\overline{F_{p_2,q}}. \end{aligned}$$

Thus, the invariance of the combined generalized moments is shown.

Note that the generalized moment $F_{p,q}$ is rewritten as

$$F_{p,q} = \int S_q(r)g_p(r)dr, \tag{6}$$

where

$$S_q(r) = \int f(r\cos\theta, r\sin\theta)\exp(\sqrt{-1}q\theta)d\theta. \tag{7}$$

By this expression of $F_{p,q}$, we have the followings.

1. Setting $g_p(r) = r^p$ with some constraints on p and q, we obtain Hu's moments [5] and Li's moments [6].
2. Setting $g_p(r)$ to be the following orthogonal polynomials:

   $$g_p(r) = \sum_{s=0}^{(p-|q|)/2} (-1)^s \frac{(p-s)!\, r^{p/2-s}}{s!(\frac{p+|q|}{2}-s)!(\frac{p-|q|}{2}-s)!},$$

   we have Zernike's moment invariants $||F_{p,q}^{Zernike}||$.

Note that Shen and Ip [7] used this technique further to construction of invariant wavelet moments.

3. Lie algebra methods for invariants

In this section we briefly review the Lie algebra method summarized by Diami [1]. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an image (usually $n = 2$ or 3), and $T_{\boldsymbol{\beta}}$ be a transformation parameterized by a vector $\boldsymbol{\beta} = (\beta_1, ..., \beta_\ell)$. A functional M is said to be invariant under the transformation $T_{\boldsymbol{\beta}}$ if and only if

$$M(f(\boldsymbol{x})) = M(f(T_{\boldsymbol{\beta}}(\boldsymbol{x}))) \text{ for all } \boldsymbol{\beta}, \tag{8}$$

where $\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{R}^n$. In image analysis, the transformation $T_{\boldsymbol{\beta}}$ usually represents the Lie-group transformation. Then, the framework of general moment invariants can be applied to find invariants M which are functions of regular moments in the n-dimensional case.

Definition 3.1. A vector $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ with α_i being a non-negative integer is said to be a multi-index and the order of $\boldsymbol{\alpha}$ is defined as $d = \sum_{i=1}^n \alpha_i$.

Definition 3.2. A monomial of a vector $\boldsymbol{x} = (x_1, ..., x_n)$ with a multi-index $\boldsymbol{\alpha}$ is defined by $\boldsymbol{x}^{\boldsymbol{\alpha}} = \prod_{i=1}^n x_i^{\alpha_i}$.

Definition 3.3. The regular moment of the image $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$m_{\boldsymbol{\alpha}} = m_{\boldsymbol{\alpha}}(f) = \int \cdots \int_{\mathbb{R}^n} \boldsymbol{x}^{\boldsymbol{\alpha}} f(\boldsymbol{x})d\boldsymbol{x}. \tag{9}$$

Definition 3.4. Let $\mathbb{R}^e$ be the regular moment space where $e = e(d)$ denotes the dimension of moments up to the degree d. Further define the regular moment vector $\boldsymbol{m}$ by $\boldsymbol{m} = (m_{\boldsymbol{\alpha}}) : e \times 1$. Then, the $\phi_{\boldsymbol{\beta}}$ is defined by the function from $\mathbb{R}^e \to \mathbb{R}$ induced by the Lie-transformation such that

$$\phi_{\boldsymbol{\beta}}(\boldsymbol{m}(f(\boldsymbol{x})) = \boldsymbol{m}(f(T_{\boldsymbol{\beta}}(\boldsymbol{x}))). \tag{10}$$

We can assume that ϕ is a linear function, that is,

$$\phi_{\boldsymbol{\beta}}(\boldsymbol{m}) = A_{\boldsymbol{\beta}}\boldsymbol{m} \tag{11}$$

where $A_{\boldsymbol{\beta}}$ is an appropriate $e \times e$ matrix.

Definition 3.5. The infinitesimal transformation is defined by

$$U_i = \frac{\partial A_{\boldsymbol{\beta}}}{\partial \beta_i}\Big|_{\boldsymbol{\beta}=0} \text{ for } i = 1, ..., \ell, \tag{12}$$

where $T_{\boldsymbol{\beta}}$ is parameterized such that T_0 is the identity transformation.

Proposition 2 (Diami [1]). The infinitesimal change of the moment vector $\boldsymbol{m}$ induced by the infinitesimal change of $\boldsymbol{\beta} = (\beta_1, ..., \beta_\ell)$ is

$$d\boldsymbol{m} = \left(\sum_{i=1}^{\ell} U_i d\beta_i\right)\boldsymbol{m}.$$

The goal is to derive the functional M satisfying

$$M(\boldsymbol{m}) = M(\phi_{\boldsymbol{\beta}}(\boldsymbol{m})). \tag{13}$$

This is expressed by the increment of M with respect to $\boldsymbol{\beta}$ is zero vector, that is,

$$dM = 0.$$

Since

$$dM = \langle \nabla M, d\boldsymbol{m} \rangle,$$

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we have

$$dM = \Big\langle \nabla M, \Big(\sum_{i=1}^{\ell} U_i d\beta_i\Big) d\boldsymbol{m} \Big\rangle = \sum_{i=1}^{\ell} \langle \nabla M, U_i \boldsymbol{m} \rangle d\beta_i$$
$$= \sum_{i=1}^{\ell} L_i M d\beta_i = 0 \qquad (14)$$

where $\nabla = (..., \partial/\partial m_{\boldsymbol{\alpha}}, ...)^t$ denotes the nabla operator and $L_i = \big\langle U_i \boldsymbol{m},\ \partial/\partial \boldsymbol{m} \big\rangle$. From this, $dM = 0$ is equivalent to

$$L_i M = 0 \quad \text{for} \quad i = 1, 2, ..., \ell.$$

The above argument suggests the following process of constructing an invariant function M of regular moments $\boldsymbol{m}_{\boldsymbol{\alpha}}$.

1. Find the matrix $A_{\boldsymbol{\beta}}$ satisfying the condition: $\boldsymbol{m}(f(T_{\boldsymbol{\beta}}(\boldsymbol{x}))) = A_{\boldsymbol{\beta}} \boldsymbol{m}(f(\boldsymbol{x}))$.
2. Calculate the infinitesimal generator $L_i = \big\langle U_i \boldsymbol{m},\ \partial/\partial \boldsymbol{m} \big\rangle$ for $i = 1, 2, ..., \ell$ where the matrix U_i is defined by the formula (12).
3. Solve the system of partial differential equations $L_i M = 0$, $i = 1, 2, ..., \ell$.

See Diami [1] in detail.

4. New series of invariant moments

We restrict our attention to 2-dimensional images ($n = 2$). Along the argument in Section 3, we have the following series of rotation invariants:

Theorem 1. Let

$$M_d = \sum_{i=0}^{d} \binom{d}{i} m_{i,d-i}^2 \quad \text{for} \quad d = 2, 3, ... \qquad (15)$$

be a function of the regular moments $m_{p,q}$ defined by (1). Then, the function M_d is rotation invariant.

Proof. Since the rotation group on $\mathbb{R}^2$ is a one parameter group, we can omit the suffix from U_i defined by (12) and denote it simply by U. Let

$$\boldsymbol{m}_d = (m_{d,0}, m_{d-1,1}, ..., m_{0,d})^t : (d+1) \times 1 \qquad (16)$$

be a vector consisting of all moments with order d. Since the formula M_d of (15) is a function of the vector $\boldsymbol{m}_d$, we only consider the action on $\boldsymbol{m}_d$. Now, we must show that $LM_d = 0$. For showing the relation, it is sufficient to show that $\big\langle U\boldsymbol{m}_d, \partial M_d/\partial \boldsymbol{m}_d \big\rangle = 0$.

We consider the regular moment of the rotated image with angle β, say $m'_{p,q}$. Then, it holds that

$$\begin{aligned} & m'_{p,q} \\ = & \iint (x\cos\beta - y\sin\beta)^p \\ & \times (x\sin\beta + y\cos\beta)^q f(x,y)dxdy \\ = & \iint \left\{ \sum_{r=0}^{p} \binom{p}{r} (x\cos\beta)^r (-y\sin\beta)^{p-r} \right\} \\ & \times \left\{ \sum_{s=0}^{q} \binom{q}{s} (x\sin\beta)^s (y\cos\beta)^{q-r} \right\} f(x,y)dxdy \\ = & \sum_{r=0}^{p}\sum_{s=0}^{q} (-1)^{p-r} \binom{p}{r}\binom{q}{s} m_{r+s,p+q-r-s} \\ & \times (\sin\beta)^{p-r+s} (\cos\beta)^{q+r-s}. \end{aligned} \qquad (17)$$

In differentiation $dm'_{p,q}/d\beta\big|_{\beta=0}$ of the moment (17), only two terms with $(r,s) = (p,1)$, $(p-1,0)$ survive and others vanish.

Now, consider the case $q = d-p$ with fixed d, and let $U : (d+1)\times(d+1)$ be a matrix defined by (12) in this case. Then, the row and column of U are corresponding to the moments $\boldsymbol{m}'_d = (m'_{d,0}, m'_{d-1,1}, ..., m'_{0,d})^t$ and $\boldsymbol{m}_d$ defined by (16) respectively. Thus, elements of the matrix U are indexed by a set of numbers $\{0, 1, ..., d\}$. As a consequence of the discussion following the formula (17), the matrix U has only non-zero elements at positions $(p, p+1)$ and $(p+1, p)$ for $p = 0, 1, ..., d$. Then, it holds that

$$\begin{aligned} & \langle U\boldsymbol{m}_d,\ \partial M_d/\partial \boldsymbol{m}_d \rangle \\ & = 2\sum_{p=0}^{d} \{(d-p)m_{p+1,d-p} - p\, m_{p,d-p+1}\} \binom{d}{p} m_{p,d-p} \end{aligned} \qquad (18)$$

and it is easy to check that the quantity vanishes. This shows $LM_d = 0$, and completes the proof.

Next we have another series of new invariants with free parameters.

Theorem 2. Let M be a quadratic form of the moment vector $\boldsymbol{m}_d$ defined by (16). If the form M satisfies the relation: $\partial M/\partial \boldsymbol{m}_d = V\boldsymbol{m}_d$ with constant matrix V, the equation $LM = 0$ (rotation invariant) holds if and only if

$$U^t V + V^t U = \boldsymbol{O}. \qquad (19)$$

Further, V is a matrix solution of the equation if and only if V is a symmetric matrix expressed by $V = (U^{-1})^t W$ with W being some alternative matrix.

Proof. Under the setup in Theorem 2, we have

$$\langle U\boldsymbol{m}_d, \partial M/\partial \boldsymbol{m}_d\rangle = \langle U\boldsymbol{m}_d, V\boldsymbol{m}_d\rangle = \boldsymbol{m}_d^t U^t V \boldsymbol{m}_d.$$

Hence,

$$d\langle U\boldsymbol{m}_d, V\boldsymbol{m}_d\rangle = 0$$

if and only if U^tV is an alternative matrix satisfying the equation (19). Further, the matrix V should be symmetric because $\partial^2 M/\partial \boldsymbol{m}_d\, \partial \boldsymbol{m}_d' = V$. If a symmetric matrix V satisfies the equations $\partial M/\partial \boldsymbol{m}_d = V\boldsymbol{m}_d$ and (19), the function M would be rotation invariant.

A. Illustration of Theorem 2

We will derive invariants by Theorem 2 in the following cases with $d = 2, 3, 4$.

A.1 The case $d = 2$

It follows that

$$M = \frac{a}{2}m_{20}^2 + \frac{b}{2}m_{11}^2 + \frac{a}{2}m_{02}^2 + (a - \frac{b}{2})m_{20}m_{02} \quad (20)$$

$$= \frac{a}{2}(m_{20} + m_{02})^2 - \frac{b}{2}(m_{20}m_{02} - m_{11}^2) \quad (21)$$

is rotation invariant for the case of $d = 2$, where a and b are arbitral real numbers. Note that neither $m_{20} + m_{02}$ nor $m_{20}m_{02} - m_{11}^2$ cannot be derived by Theorem 1. We omit the proof.

A.2 The case $d = 3$

Set $\boldsymbol{m} = \boldsymbol{m}_3 = (m_{30}, m_{21}, m_{12}, m_{03})^t$. Then, the matrix U defined by (12) is given by

$$U = \begin{pmatrix} 0 & -3 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 3 & 0 \end{pmatrix},$$

and the inverse matrix is given by

$$U^{-1} = \frac{1}{3}\begin{pmatrix} 0 & 3 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \end{pmatrix}.$$

Put

$$W_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

as a matrix of base of alternative matrices. Then, $V_1 = (U^{-1})^t W_1$ is given by

$$V_1 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

In the same way, for

$$W_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ W_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$W_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ W_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$W_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

we have $V_i = (U^{-1})^t W_i$ as

$$V_2 = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 \end{pmatrix},\ V_3 = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$V_4 = \frac{1}{3}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ V_5 = \frac{1}{3}\begin{pmatrix} 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_6 = \frac{1}{3}\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\quad \text{respectively.}$$

Unfortunately, these V_i do not correspond to any function M satisfying $\partial M/\partial \boldsymbol{m} = V_i\boldsymbol{m}$. So, we must find a function M whose derivatives $\partial M/\partial \boldsymbol{m}$ is expressed by a linear combination $V = \sum_{i=1}^{6} \alpha_i V_i$ because any alternative matrix W is a linear combination of W_i and the corresponding V is also a linear combination of V_i. It is shown that V must be symmetric if the equation $\partial M/\partial \boldsymbol{m} = V\boldsymbol{m}$ holds. The symmetry of the matrix V leads immediately that the coefficients α_2 and α_5 equal to zero. After some calculation, it is shown that V is of the form

$$V = \frac{1}{3}\begin{pmatrix} \alpha_1 + 2\alpha_3 & 0 & 3\alpha_3 & 0 \\ 0 & 3\alpha_1 & 0 & 3\alpha_3 \\ 3\alpha_3 & 0 & 3\alpha_1 & 0 \\ 0 & 3\alpha_3 & 0 & \alpha_1 + 2\alpha_3 \end{pmatrix}.$$

Also, any alternative matrix W is expressed in a linear combination of $W_1, ..., W_6$ and the corresponding invariant is a linear combination of these $V_i, i = 1, 2, ..., 6$.

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Hence we have the invariant:

$$\begin{aligned} M(\alpha_1,\alpha_3) &= \frac{1}{2}(\frac{\alpha_1}{3}+\frac{2\alpha_3}{3})m_{30}^2 \\ &+ \alpha_3 m_{30}m_{12}+\frac{\alpha_1}{2}m_{21}^2+\alpha_3 m_{21}m_{03} \\ &+ \alpha_3 m_{30}m_{12}+\frac{1}{2}\alpha_1 m_{12}^2+\alpha_3 m_{21}m_{03} \\ &+ \frac{1}{2}(\frac{\alpha_1}{3}+\frac{2\alpha_3}{3})m_{03}^2 \qquad (22)\end{aligned}$$

where α_1 and α_3 is arbitral real numbers. Actually, it holds that $M(\alpha_1,\alpha_3)=\alpha_3 M_{3,1}/6+\alpha_3 M_{3,2}/6$, where

$$\begin{aligned} M_{3,1} &= m_{30}^2+3m_{21}^2+3m_{12}^2+m_{03}^2, \qquad (23)\\ M_{3,2} &= m_{30}^2+3m_{30}m_{12}+3m_{03}m_{21}+m_{03}^2. \qquad (24)\end{aligned}$$

The invariant $M_{3,1}$ in (22) is already given by Theorem 1, whereas $M_{3,2}$ is not.

A.3 The case $d=4$

In this case we must obtain a pair of matrices (V,W) such that V and W are respectively symmetric and alternative matrices satisfying $U^tV=W$, where

$$U^t=\begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ -4 & 0 & 2 & 0 & 0\\ 0 & -3 & 0 & 3 & 0\\ 0 & 0 & -2 & 0 & 4\\ 0 & 0 & 0 & -1 & 0\end{pmatrix}.$$

After some elementary calculation, the general solution is given by

$$V=\begin{pmatrix} a & 0 & 2a-b/2 & 0 & a-b/2\\ 0 & b & 0 & b & 0\\ 2a-b/2 & 0 & 4a-b & 0 & 2a-b/2\\ 0 & b & 0 & b & 0\\ a-b/2 & 0 & 2a-b/2 & 0 & a\end{pmatrix},$$

and

$$W=b\begin{pmatrix} 0 & 1 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 1\\ 0 & -1 & 0 & -1 & 0\end{pmatrix},$$

where a and b are arbitral real numbers. Hence the invariant with two parameters is given by

$$\begin{aligned} M(a,b) &= \frac{a}{2}m_{40}^2+(2a-\frac{b}{2})m_{40}m_{22} \\ &+ (a-\frac{b}{2})m_{40}m_{04}+\frac{b}{2}m_{31}^2+bm_{31}m_{13} \\ &+ (2a-\frac{b}{2})(m_{40}m_{22}+m_{22}^2+m_{22}m_{04}) \\ &+ bm_{31}m_{13}+\frac{b}{2}m_{13}^2+(a-\frac{b}{2})m_{40}m_{04} \\ &+ (2a-\frac{b}{2})m_{22}m_{04}+\frac{a}{2}m_{04}^2 \qquad (25)\\ &= \frac{a}{2}M_{4,1}-\frac{b}{2}M_{4,2} \qquad (26)\end{aligned}$$

where

$$\begin{aligned} M_{4,1} &= m_{40}^2+8m_{40}m_{22}+4m_{40}m_{04}+4m_{22}^2 \\ &+ 8m_{22}m_{04}+m_{04}^2, \qquad (27)\\ M_{4,2} &= 2m_{40}m_{22}+2m_{40}m_{04}+2m_{22}m_{04}+m_{22}^2 \\ &- m_{31}^2-4m_{31}m_{13}-m_{13}^2. \qquad (28)\end{aligned}$$

Note that Theorem 1 does not give the invariants $M_{4,1}$ nor $M_{4,2}$.

5. Conclusion

For 2-D images, we derived two special families of rotation invariant moments by using the Lie group theory. One of the families is an infinite sequence of the squared sum of regular moments, and the other is given by the solution of matrix equations. Both families are, as far as the authors know, have not been stated explicitly as a family of rotation invariant moments.

The performance of the proposed families will be examined through the recognition of handwritten words of old Syriac "Estrangelo" somewhere.

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